The Principle of Excluded Middle

I. Lesson

"In regard to things present or past, propositions, whether positive or negative, are true of necessity or false. And of those contradictorily opposed one, again, must be true and one false [...]."
-Aristotle, *On Interpretation* 9 (Harold P. Cook, 1962)

Previously, we discussed a particular kind of proposition pair: *contradictories*. There are two interesting basic principles that characterize certain relations between such propositions: *the Principle of Excluded Middle* (PEM, for short) and *the Principle of Non-Contradiction* (PNC, for short). We have already talked about the latter. Let's turn to the former.

This basic logical principle expresses a fairly intuitive fact – for any pair of contradictory statements, at least one must be true. For any proposition, *P*, and its contradictory, not-*P*, the formal statement of PEM is this:

- **[PEM]:** Either *P* or not-*P*, but not both, must be true.

To make this more specific, recall our earlier lesson on the square of opposition. PEM justifies our statement of the *contradictory opposition* between Universal Affirmative and Particular Negative propositions, or between Universal Negative and Particular Affirmative propositions. By our earlier definitions, the truth of a Universal Affirmative proposition implies the falsity of a Particular Negative proposition. We can stress this as a conditional statement to make the PEM more precise: *if* a Universal Affirmative proposition is true, *then* its correlated Particular Negative proposition is false. Note that the PEM does *not* target specific individual propositions, but *pairs of contradictory opposites* on the square of opposition.

[Activity] On the face of it, PEM is very intuitively reasonable. To illustrate, suppose we ask someone the question "Is chalk white?" If they answer, "Either chalk is white, or it isn't," we would probably roll our eyes. This answer is worthy of an eye-roll precisely because the PEM is so obvious to us: of course it is one or the other! Can you think of what chalk would look like if it were neither white, nor not white? Is this even possible? Discuss with a partner!

[Activity] Pair up with a classmate and discuss the following "proof" – Either the Earth is flat, or it isn't. If the Earth were flat, a large amount of our verified and independently valid scientific observations (and the theories to which they belong) would completely fall apart. But our scientific observations and the theories to which they belong are sound and reliable. Therefore, the Earth isn't flat. How is PEM being used here?

II. Examples

Let's look at another step-by-step example of PEM at work to further emphasize just how intuitively basic it appears to us in common cases. This time around, let's look at a proposition with the verb 'to be' in the past tense to illustrate that PEM holds in straightforward propositions about the past as well.

1. "Either dinosaurs were large on average, or they weren't."

In the case of past events, as in the case of present ones, evaluating the truth of 1. is a matter of checking against the facts of the world to see if there is a correspondence between what was/is the case and the content of the proposition. It seems trivially true that either dinosaurs were large on average, or they weren't – whatever the relevant state of affairs indicates will make one of the two disjuncts true; therefore, the entire expression, "Either dinosaurs were large on average, or they weren't." will be true by virtue of one of the contradictories in the pair being true. So far, so good. What PEM is saying further is that *generally*, for *any* pair of contradictories, this result holds.

2. "All dinosaurs were large."

The above example is a Universal Affirmative proposition. Clearly enough, it is false - there were small dinosaurs. By the square of opposition, the falsehood of example 2 implies the truth of its contradictory opposite, namely that "Some dinosaurs are not large." This is obvious enough on the face of it, but we might reasonably ask a further question: why should we believe that the implication holds? Put otherwise, why should we think that the relation of contradictory opposition works in this way? The answer: PEM. The principle gives us reason to believe - it *justifies* - that contradictory opposition works exactly as we expect it to.

III. Exercises

Exercise 1

Explain the difference between the principle of excluded middle and the principle of bivalence (recall that they are closely connected. One is about pairs of contradictory opposites, the other is about individual propositions).

Exercise 2

In the following sketch proofs, mark by what principle (PEM, PNC, or the principle of identity) each of the **bolded** steps follows (Do not worry over whether or not the proofs are certainly sound).

e.g.: Apples are not oranges.

Proof:

a) **Either apples are oranges, or apples are not oranges.** (by PEM)
b) Suppose apples are oranges.
c) **Then, all apples must taste like oranges.** (by the principle of identity)
d) But no apple tastes like an orange.
e) **Therefore, apples are not oranges.** (by PNC, b) leads to a contradiction, so it must be false)

i. There is no largest number.

Proof:

a) **Either there is a largest number, or there is no largest number.**
b) Suppose there is a largest number, L.
c) Then, there is no number that is larger than L.
d) We can add numbers together.
e) **Then, there is a number Q that is equal to $L + 1$.**
f) But then, there exists a number Q that is larger than L.
g) **Therefore, there is no largest number.**

ii. PEM is true.

Proof:

a) **Either PEM is true, or PEM is not true.**
b) Suppose PEM is not true.
c) Then, all instances of PEM are not true.
d) Then, "Either PEM is true, or PEM is not true" is itself not true.
e) Then, PEM is true and PEM is not true simultaneously.
f) But e) is a contradiction.
g) **Therefore, PEM is true.**

iii. Apples are nutritious.

Proof:

a) **Food is nutritious if and only if it somehow promotes health.**
b) **Foods that contain healthy nutrients are things that promote health.**
c) **Either apples promote health, or apples do not promote health.**
d) Apples are a kind of food that contains healthy nutrients.
e) Therefore, apples promote health.
f) Therefore, apples are nutritious.

Exercise 3

(Challenge) Think of *proofs by contradiction* more generally, or what Aristotle calls "deductions through impossibility". How is PEM used in making such proofs? How is PNC used? [Hint: There might be an implicit assumption that we make right before we negate the conclusion and try to deduce a contradiction from it].

Exercise 4

State whether the following examples are instances of PEM, the principle of bivalence, or neither (i.e. they are merely assertions).

e.g. "The sky is blue" is either true, or false – this is an instance of the principle of bivalence, because it is a statement about the truth-value of an individual assertion.

i. Either the sky is blue, or the sky is not blue.
ii. There are two possible truth-values for "The sky is blue."
iii. The sky is either blue, or not blue.
iv. There is a unique truth-value – truth or falsehood – for "The sky is blue."
v. "Either the sky is blue, or the sky is not blue" is true.
vi. "Either the sky is blue, or the sky is not blue" is either true, or false.

Exercise 5

(Extra Challenge) Recall *double negation*, the logical principle which states that any proposition is equivalent to itself negated twice, from our introductory lesson of propositions. For example, "Chalk is white." is equivalent to "It is not the case that chalk is not white." and "All trees are human." is equivalent to "It is not the case that all trees are not human." In terms of truth-value, negation works as a switch, a toggle: if we take a true proposition and negate it once, we get a false one; if we negate that again, we get a true proposition. Use double negation to get from PEM to PNC [Hint: think about what happens if you negate PEM twice and look at how we would assign truth-values to the "nested" propositions].

IV. Conclusion

The principle of excluded middle is one of the foundational principles of Aristotelian logic, widely employed in modern logic, and present, whether we notice it or not, in our day-to-day reasoning.

In this course, we have covered some of the basic principles of Aristotelian Logic, from his analysis of terms, to his categorical propositions, to his syllogistic logic. Along the way, we touched on topics and methods not mentioned by Aristotle, but those which are immensely helpful in the study of modern logical analysis and interpretation. Ultimately, the totality of logic as we know it has built in some way or another upon many of the core principles of Aristotle.

V. Lesson Goals

At the end of this lesson, students will be able to:

- define the principle of excluded middle;
- differentiate PEM from the principle of bivalence;
- characterize the apparent intuitive "givenness" of PEM;
- characterize the importance of PEM to *reductio ad absurdum* proofs;
- (optional) roughly characterize the equivalence of PEM and PNC as tautologies;

Answer key:

Exercise 1

The principle of excluded middle targets pairs of contradictory opposites, stating that at least one of them must be true. The principle of bivalence targets individual propositions, stating that they must be either true, or false.

Exercise 2

i. a) by PEM; e) by the principle of identity; g) by PNC; ii. a) by PEM; g) by PNC; iii. a) by the principle of identity; b) by the principle of identity; c) by PEM.

Exercise 3

We leverage both PEM, and PNC in a proof by contradiction. If we have to prove *P* by contradiction, the first step is noting that either *P*, or not-*P* – PEM itself. If assuming not-*P* leads to a contradiction, then, by PNC, not-*P* is false. By PEM, since at least one proposition must be true, it follows that *P* must be true.

Exercise 4

i. PEM; ii. neither; iii. PEM; iv. the principle of bivalence; v. neither; vi. the principle of bivalence.

Exercise 5

Negating PEM twice, we obtain "It is not the case that it is not the case that either *P*, or not-*P*." It must be that "It is not the case that either *P*, or not-*P*" is false, so that negating it again yields a truth. For any truth-value of *P*, one of the disjuncts will be true and the other false, so the disjunction will be true and its negation will always be false. In other words, this amounts to saying "Both *P*, and not-*P*.", a statement which is always false for every truth-value of *P*. Putting this together with our previous result, we obtain "It is not the case that both *P*, and not-*P*.", which is the given form of the PNC.